

# The Network Structure of Mathematical Knowledge According to the Wikipedia, MathWorld, and DLMF Online Libraries

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## Abstract

We study the network structure of Wikipedia (restricted to its mathematical portion), MathWorld, and DLMF. We approach these three online mathematical libraries from the perspective of several global and local network-theoretic features, providing for each one the appropriate value or distribution, along with comparisons that, if possible, also include the whole of the Wikipedia or the Web. We identify some distinguishing characteristics of all three libraries, most of them supposedly traceable to the libraries' shared nature of relating to a very specialized domain. Among these characteristics are the presence of a very large strongly connected component in each of the corresponding directed graphs, the complete absence of any clear power laws describing the distribution of local features, and the rise to prominence of some local features (e.g., stress centrality) that can be used to effectively search for keywords in the libraries.

**Keywords:** Online mathematical libraries, Wikipedia, MathWorld, DLMF, complex networks, text search.

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# 1 Introduction

Until a few decades ago, before it became commonplace to search the Web for information and knowledge, people desiring quick access to some mathematical concept or formula used to resort to printed encyclopedias or handbooks, such as the compilation by Abramowitz and Stegun [1], the more specialized tables put together by Gradshteyn and Ryzhik [26], or still others [28, 15]. Of such volumes, the undisputed citations champion seems to be Abramowitz and Stegun’s [8], whose work has since been methodically expanded [32] into a NIST-sponsored publication [41].

Lately, though, the situation has become, if anything, more complex. For, while those printed works continue to be used and cited widely and their ranks continue to be enlarged by the addition of new works of a similar genre [25], the premier source, at least for a first approach, has undoubtedly become the Web. In fact, it seems safe to state that most mathematics-related queries on Google return Wikipedia<sup>1</sup> or Wolfram MathWorld<sup>2</sup> pages as prominently ranked. As mentioned, however, printed and online material still coexist and, curiously, movement has taken place in both directions: while in one direction MathWorld material has found its way into Weisstein’s encyclopedia [47], in the other the NIST volume has been turned into the Digital Library of Mathematical Functions, DLMF.<sup>3</sup>

Here we aim to explore the structure of mathematical knowledge as reflected in these three online libraries. By “structure” we do not mean the organization of material into the many mathematical areas and subareas. Nor do we mean the coalescence of all deduction chains that is behind all of mathematics and inherently amounts to an acyclic directed graph [17], i.e., one with no directed cycles. We mean, rather, the no longer acyclic directed graphs that reflect all the cross-referencing that took place as those libraries were created by several collaborators (and still takes place as the libraries evolve). Exploring their graph structures from the perspective of such hypertextual interconnections amounts to applying some of the complex-network notions and metrics developed during the past fifteen years or so, much as has been done so successfully to various other fields [11, 35, 10].

It also amounts to a chance to globally view all the material compiled into each library and inquire, from a network-theoretic perspective, what traces remain, if any, as telltale signs of the essentially very distinct methods of construction employed to build them, all of a collaborative nature but supposedly more and more controlled as we move from Wikipedia to MathWorld and then to DLMF. In our analyses we use several frequency data, of both a network-wide nature as well as node-related, aiming not only to describe the libraries’ properties as such data reveal them, but also to discover how these properties relate to the libraries’ robustness in the face of accidental or intentional loss of material and to their ease of search in response to text queries.

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<sup>1</sup><http://en.wikipedia.org/wiki/Portal:Mathematics>.

<sup>2</sup><http://mathworld.wolfram.com>.

<sup>3</sup><http://dlmf.nist.gov>.

What has turned up is a collection of results that both sets the three mathematical libraries apart from the wider English-language Wikipedia and from the much wider Web, and at the same time groups the three libraries together insofar as they share important properties. Some of the most significant results include the characteristic that a very large fraction of each library’s pages are packed together in the sense of mutual reachability; the presence of clear signs that all three libraries result from decisions regarding the deployment of links that are leveraged by technical knowledge (rather than, say, some nontechnical measure of a page’s relevance, such as popularity); and the discovery of successful criteria for guiding text search within the libraries’ pages that differ significantly from those most commonly used (e.g., by Google).

We proceed in the following manner. First, in Section 2, we introduce the five directed graphs that we use in all analyses (two for Wikipedia, two for MathWorld, one for DLMF) and also some basic notation. We then move, respectively in Sections 3 and 4, to a study of these graphs’ global and local network-theoretic features. Section 5 is dedicated to an analysis of the five graphs’ robustness when nodes are lost either as a result of some random process or as a deterministic function of the graphs’ local features. We continue with Section 6, where we investigate the effect of such features in the ranking of nodes when responding to text queries. We conclude in Section 7.

## 2 Five directed graphs

In all three libraries it is possible to reach the technical-content pages by navigating through a hierarchy of specialized subdivisions from the main portal (the so-called category pages). Once the content pages are reached, further navigation is possible through the links that lead from one such page to another. Each of the directed graphs with which we work has a node for each content page and directed edges that reflect inter-page links. In all cases, links leading from a page to itself are ignored when building the graph, so no self-loops exist. Similarly, should multiple links exist from a page to another, only one edge is created in the graph between the corresponding nodes.

In the case of Wikipedia and MathWorld, links can be categorized into those appearing in a page’s main text and those that are given in the page’s “See also” section when it exists. We perceive these two link types as playing entirely different roles. While in-text links are generally meant to clarify some of the terms used in the page, being therefore meant for quick side lookups before continuing on the main text, See-also links are used to point to pages where related material is to be found. For this reason, we use two different graphs for each of Wikipedia and MathWorld. They both have the same node set, but their edge sets differ, one reflecting in-text as well as See-also links, the other reflecting See-also links only.

The case of DLMF requires no such special treatment. Although its pages, too, contain special, “Referenced by” links, such links are simply antiparallel versions of the library’s non-Referenced-by links. That is, page  $a$  contains a

Table 1: Online libraries and corresponding directed graphs.

Library	Download period	Directed graph
Wikipedia	September 2010	$W$
Wikipedia, See-also links	September 2010	$W'$
MathWorld	August 2009	$M$
MathWorld, See-also links	August 2009	$M'$
DLMF	September 2010	$D$

non-Referenced-by link to page  $b$  if page  $b$  contains a Referenced-by link to page  $a$ . Referenced-by links in DLMF are therefore redundant as far as building its directed graph is concerned. They are for this reason ignored.

These observations amount to five different graphs with which to work, as summarized in Table 1. In the table, for each of the libraries and, when applicable, taking See-also links into account, we give the time frame within which the content pages were downloaded and the notation we use to refer to the corresponding graph.

Some additional basic notation to be used throughout is the following. Given the graph under consideration, we let  $n$  stand for its number of nodes and  $m$  for its number of edges. For node  $i$ ,  $I_i$  is its set of in-neighbors (nodes from which edges are directed toward  $i$ ) and  $O_i$  its set of out-neighbors (nodes toward which edges are directed from  $i$ ). Its in-degree is  $\delta_i^+ = |I_i|$ , its out-degree is  $\delta_i^- = |O_i|$ , and its number of neighbors when edge directions are disregarded (henceforth referred to simply as its degree) is  $\delta_i = |I_i \cup O_i| \leq \delta_i^+ + \delta_i^-$ . Clearly, it holds that  $\max\{\delta_i^+, \delta_i^-\} \leq \delta_i$ . For any two nodes  $i$  and  $j$ ,  $d_{ij}$  is the distance from  $i$  to  $j$ , that is, the number of edges on a shortest directed path leading from  $i$  to  $j$ . If none exists, then  $d_{ij} = \infty$ . We let  $R_i$  be the set of nodes  $j$  such that  $0 < d_{ij} < \infty$ . Note that  $R_i = \emptyset$  if and only if node  $i$  is a sink, i.e.,  $O_i = \emptyset$ .

### 3 Global features

We give six global features for each graph. The first two are straightforward and provide simple relationships between the graph's number of nodes,  $n$ , and its number of edges,  $m$ . The first one is simply the graph's mean in-degree, denoted by  $\delta^+$  and given by

$$\delta^+ = \frac{1}{n} \sum_i \delta_i^+ = \frac{m}{n} \quad (1)$$

(necessarily equal to the graph's mean out-degree). The second feature is the graph's mean degree. Denoting it by  $\delta$ , we have

$$\delta^+ \leq \delta = \frac{1}{n} \sum_i \delta_i \leq \frac{1}{n} \sum_i (\delta_i^+ + \delta_i^-) = 2\delta^+. \quad (2)$$

Both  $\delta^+$  and  $\delta$  work as indicators of the graph's edge density relative to its number of nodes. The value of  $\delta$ , in particular, may swing toward either of its

bounds,  $\delta^+$  and  $2\delta^+$ , indicating in the former case that every edge's antiparallel counterpart is also present in the graph and in the latter case that none is. On average, then, the fraction of  $\delta$  corresponding to antiparallel edge pairs is given by  $(2\delta^+ - \delta)/\delta = 2\delta^+/\delta - 1$ .

Our next global feature is the fraction  $S$  of  $n$  that corresponds to the nodes inside the graph's largest strongly connected component (GSCC henceforth, where  $G$  is for "giant"). A strongly connected component is either a singleton whose only member, say node  $i$ , is such that  $i \notin R_j$  for every node  $j \in R_i$  (no directed path exists back from any node that can be reached from  $i$  through a directed path), or a larger set that is maximal with respect to the property that  $j \in R_i$  for any two of its members  $i$  and  $j$  such that  $j \neq i$ . In the latter case, then, a directed path exists between any two distinct nodes inside the strongly connected component. Informally, the value of  $S$  can be regarded as an indication of the network's "degree of acyclicity." If the graph is acyclic, then all its strongly connected components are singletons and  $S = 1/n$ . The other extreme corresponds to the case in which all nodes are in the GSCC, so  $S = 1$ .

The fourth and fifth global features are both related to classifying a graph vis-à-vis the so-called small-world criteria [46, 2], namely small distances and large transitivity. We address the first criterion by computing the average distance between any two distinct nodes, so long as only finite distances are considered. We denote this average by  $\ell$ , which is then such that

$$\ell = \frac{1}{N} \sum_i \sum_{j \in R_i} d_{ij}, \quad (3)$$

where  $N$  is the number of  $i, j$  pairs contributing to the double summation. As for the second criterion, that of transitivity, we follow the usual trend of disregarding edge directions and computing the resulting graph's clustering coefficient in its most common formulation [39]. If  $C$  is the clustering coefficient, then this formulation lets  $C = 3t/T$ , where both  $t$  and  $T$  refer to node triples in the graph, e.g.,  $i, j, k$ . The value of  $t$  is meant to reflect the number of triangles in the graph, that is, those triples in which an edge connects  $i$  and  $j$ , another connects  $j$  and  $k$ , and yet another connects  $i$  and  $k$ . The value of  $T$ , on the other hand, counts the triples that are arranged as three-node (two-edge) paths. The factor 3 in the numerator of the ratio defining  $C$  reflects the fact that there are three triples of the latter type for each triangle in the graph. It follows that  $0 \leq C \leq 1$  (no transitivity through full transitivity). In our analysis of each graph's clustering coefficient  $C$ , we present it side-by-side with the value it would have if every node  $i$  continued to have the same degree  $\delta_i$  but the connections were made at random [39]. This value, denoted by  $C'$ , is given by

$$C' = \frac{(\delta^{(2)} - \delta)^2}{n\delta^3}, \quad (4)$$

where  $\delta^{(2)} = (1/n) \sum_i \delta_i^2$ .

Our last global feature is in fact a series of four assortativity coefficients. Each one is the Pearson correlation coefficient of two length- $m$  sequences of numbers. If  $\alpha_1, \alpha_2, \dots, \alpha_m$  and  $\beta_1, \beta_2, \dots, \beta_m$  are the sequences,  $\mu_\alpha$  and  $\mu_\beta$  are the corresponding means, and  $\sigma_\alpha$  and  $\sigma_\beta$  are the corresponding standard deviations, this coefficient is

$$r_{\alpha,\beta} = \frac{(1/m) \sum_e \alpha_e \beta_e - \mu_\alpha \mu_\beta}{\sigma_\alpha \sigma_\beta}. \quad (5)$$

The original assortativity coefficient is obtained by letting  $\alpha_e = \delta_i^-$  and  $\beta_e = \delta_j^+$  for  $e$  the edge directed from  $i$  to  $j$  [36, 37]. That is, it measures how correlated the out-degrees of the edges' tail nodes are with the in-degrees of the edges' head nodes. A shorthand for this formulation is to use out,in in place of  $\alpha, \beta$ . We get the other three variations by selecting the other possible combinations (in,out; out,out; in,in) [23, 42].

The global features of the graphs in Table 1 are shown in Tables 2 and 3, which include an additional row for the directed graph, denoted by  $W^+$ , that corresponds to the entire English-language Wikipedia of a relatively recent past [16, 48]. Table 2, moreover, contains one further row for the whole Web, now based on data from an older past [14].<sup>4</sup> The corresponding directed graph is denoted by  $W^*$ . Not all global features are available for  $W^+$  or  $W^*$ , as indicated by blank entries in the tables. Graphs are arranged in Tables 2 and 3 in nonincreasing order of  $n$ , then in decreasing order of  $m$ .

The data shown in Table 2 indicate that edge density relative to the number of nodes, as given by  $\delta^+$ , has the same order of magnitude for most graphs, the exception being  $W'$ , the Wikipedia graph based exclusively on See-also links, whose  $\delta^+$  value is one order of magnitude lower. Wikipedia contributors to the mathematical pages, therefore, seem to deploy See-also links considerably less methodically than those who contribute to MathWorld. It is also worth noting that the five mathematics-related graphs have fairly different values for the ratio  $2\delta^+/\delta - 1$ , pointing at  $W'$  as the graph with the fewest antiparallel edge pairs contributing to degrees on average, and to  $M'$ , the MathWorld graph based on See-also links, as having the most. Once again, then, MathWorld contributors appear more meticulous at providing cross-referencing information of the See-also form.

One of the most striking contrasts in Table 2 concerns the value of  $S$ , the size of the graph's GSCC relative to  $n$ . While for the Web graph  $W^*$  our current best estimate places about 28% of the nodes inside the GSCC, for the Wikipedia graph  $W^+$  and most of the mathematical-library graphs we have been considering the GSCC encompasses substantially more nodes (between 62 and 82%). The exception, once again, occurs on account of graph  $W'$ , whose GSCC is sized at a mere 2% of the nodes, and which as we have noted is only very sparsely interconnected by the See-also links.

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<sup>4</sup>Slightly more recent data seem to indicate an  $S$  value of roughly 0.33 for a similarly sized Web [19], but no estimate is given for  $\ell$ .

Table 2: Global features: mean in- or out-degree ( $\delta^+$ ), mean degree ( $\delta$ ) and the resulting value of  $2\delta^+/\delta - 1$ , fraction of  $n$  within the GSCC ( $S$ ), average distance between distinct nodes ( $\ell$ ), and clustering coefficient ( $C$ , along with the value,  $C'$ , it would have if connections were random).

Graph	$n$	$m$	$\delta^+$	$\delta$	$2\delta^+/\delta - 1$	$S$	$\ell$	$C$	$C'$
$W^*$	203 549 046	1 466 000 000	7.20			0.28	16.18		
$W^+$	339 834	5 278 037	15.53			0.82	4.90		
$W$	37 723	688 589	18.25	30.62	0.19	0.80	4.11	0.055	$7.59 \times 10^{-4}$
$W'$	37 723	21 503	0.57	1.04	0.09	0.02	15.27	0.061	$4.96 \times 10^{-8}$
$M$	15 095	92 648	6.14	9.72	0.26	0.78	5.32	0.048	$5.18 \times 10^{-4}$
$M'$	15 095	46 965	3.11	4.45	0.40	0.62	7.45	0.093	$1.77 \times 10^{-4}$
$D$	908	7 527	8.29	12.81	0.29	0.81	3.79	0.062	0.011

Table 3: Global features: assortativity coefficients.

Graph	$r_{\text{out,in}}$	$r_{\text{in,out}}$	$r_{\text{out,out}}$	$r_{\text{in,in}}$
$W^+$	-0.150			
$W$	-0.071	0.075	-0.074	-0.022
$W'$	0.041	0.094	0.070	0.028
$M$	-0.037	-0.018	-0.015	-0.019
$M'$	-0.054	-0.031	-0.058	-0.036
$D$	-0.169	0.006	-0.053	-0.043

The remaining data in Table 2 refer to  $\ell$  and to  $C$ , a graph’s average path length (in the directed sense) and clustering coefficient (in the undirected sense), respectively. We first note that, for six of the seven graphs,  $\ell$  is proportional to  $\ln n$  by a constant of the order of  $10^{-1}$ , the exception being  $W'$ , for which the proportionality constant is roughly 1.45 (this comes from the substantially larger distances in comparison to  $W$ , as expected from the substantially lower  $\delta^+$  value). In all cases, however, distances are on average very small given the value of  $n$ , so all seven graphs qualify as small-world structures. Moreover, as is usually but not always the case [39], in all five mathematics-related graphs the value of  $C$  is noticeably larger than that of  $C'$ . In fact, except for the DLMF graph  $D$ ,  $C$  surpasses  $C'$  by a factor of at least two orders of magnitude. The construction of  $D$ , which has  $C \approx 5.64C'$ , seems to have been guided by forces that prevent the formation of triangles more than they do in the other four cases. One possible explanation is that, in comparison to Wikipedia or MathWorld, each DLMF page contains substantially more material, which in fact is reflected in the low number of nodes of graph  $D$ .

Table 3 contains all four assortativity coefficients for all five mathematics-related graphs and  $W^+$ , the unrestricted Wikipedia graph. The vast majority of all values is of the order of  $10^{-2}$  at most, being therefore sufficiently near zero for the sequences involved to be taken as uncorrelated. In general this is indicative either of a random pattern of connections (which is not the case) or that criteria for edge deployment are at work that make no reference whatsoever to in- or out-degrees (which is more plausibly the case). Curiously, though, the same holds also for the only two exceptions,  $W^+$  and  $D$ , for which the moderately negative but nonnegligible value of  $r_{\text{out,in}}$  is suggestive that in these two graphs connections are effected in such a way that promotes a small but noticeable degree of disassortative mixing of tail nodes’ out-degrees with head nodes’ in-degrees. That is, there is a slight tendency of nodes with larger (smaller) out-degrees to connect out to nodes with smaller (larger) in-degrees. This tendency is quantified very similarly by  $r_{\text{out,in}}$  for both  $W^+$  and  $D$  ( $-0.150$  in the former case,  $-0.169$  in the latter). Perhaps the aforementioned fact that the typical DLMF page contains more material than the other libraries’ pages somehow makes  $D$  resemble  $W^+$  in this one aspect.



## 4 Local features

Presenting a graph’s local features requires that we value each feature of interest for each node and then provide some probability distribution of that feature over the entire graph. In this section we work with the feature’s complementary cumulative distribution (CCD henceforth), denoted by  $F(z)$  for an admissible feature value  $z$ , which is the probability that a randomly chosen node has a feature value that surpasses  $z$ . We compute  $F(z)$  as the fraction of  $n$  representing the nodes for which the feature is valued beyond  $z$ . Clearly, if for a graph the feature in question is never valued beyond  $Z$ , then  $F(z) = 0$  for all  $z \geq Z$ .

The most widely studied local features are a node’s in-degree, out-degree, and degree. Not only have they been measured in a variety of domains, but knowledge of how they are distributed can be used in the study of many other network properties [40]. These features are the first three we study, as characterizations of in- and out-degrees have over the years led to important discoveries regarding the Web and the unrestricted Wikipedia. Specifically, we know from at least two independent sources operating on different data that in-degrees in the Web graph (our  $W^*$  graph in one case, a different version in the other) are distributed according to a power law [2, 14, 19]. That is, the probability that a randomly chosen node has in-degree  $k > 0$  is proportional to  $k^{-\alpha}$  (so the corresponding CCD is approximately proportional to  $k^{1-\alpha}$ ) for  $\alpha \approx 2.1$ . Similar power laws have also been reported for the graph’s out-degrees, but in this case there seems to be some disagreement [19]. As for the Wikipedia graph,  $W^+$ , its in-degree, out-degree, and degree distributions have all been found to follow power laws, of exponents  $-2.21$ , between  $-2.65$  and  $-2$ , and  $-2.37$ , respectively [16, 48]. Power laws are inherently scale-free [38], and their appearance in graphs such as  $W^*$  and  $W^+$  has been explained particularly well by the mechanism of edge deployment known as preferential attachment [43, 6, 7, 16].

The additional local features that we consider are the ones given in Table 4. Four of them ( $B_i$ ,  $S_i$ ,  $C_i$ , and  $G_i$ ) are measures of node  $i$ ’s centrality in the graph, being therefore related to shortest directed paths in which  $i$  participates in some way. The remaining three are related to search mechanisms on the Web. They are measures of how node  $i$  qualifies as a hub ( $y_i$ ) or an authority ( $x_i$ ) in the HITS (Hyperlink-Induced Topic Search) mechanism, and the node’s page rank ( $\rho_i$ ), which underlies Google searches.

The centrality features can be computed through variations of a well-known algorithm [12], and similarly the other three, though requiring iterative updates for convergence. In the case of the HITS-related features, first every  $x_i$  and  $y_i$  is initialized to 1. Then the  $x_i$ ’s and  $y_i$ ’s are alternately updated via the rules given in Table 4. The updating of the  $x_i$ ’s is followed by a normalization of the resulting values so that  $\sum_i x_i^2 = 1$ , which is achieved by dividing each  $x_i$  by the Euclidean norm of the vector of components  $x_1, x_2, \dots, x_n$ . The updating of the  $y_i$ ’s proceeds similarly. After convergence, all features are normalized so that  $\sum_i x_i = \sum_i y_i = 1$ . As for the page-rank feature, once again every  $\rho_i$  is initialized to 1 and the update rule given in Table 4 is iteratively applied until convergence, at which time all  $\rho_i$ ’s are normalized so that  $\sum_i \rho_i = 1$ . For the

Table 4: Additional local features for node  $i$ .  $\sigma_{jk}$  is the number of shortest directed paths that lead from  $j$  to  $k$ , while  $\sigma_{jk}(i)$  counts only the ones that go through  $i$ .

Designation	Formula	Reference(s)
Betweenness centrality	$B_i = \sum_{j \neq i} \sum_{\substack{k \neq i \\ k \in R_j}} \frac{\sigma_{jk}(i)}{\sigma_{jk}}$	[4, 24]
Stress centrality	$S_i = \sum_{j \neq i} \sum_{\substack{k \neq i \\ k \in R_j}} \sigma_{jk}(i)$	[45]
Closeness centrality	$C_i = \begin{cases} \frac{1}{\sum_{j \in R_i} d_{ij}}, & \text{if } R_i \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$	[44]
Graph centrality	$G_i = \begin{cases} \frac{1}{\max_{j \in R_i} d_{ij}}, & \text{if } R_i \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$	[27]
HITS update rule for hubs	$y_i := \sum_{j \in O_i} x_j$	[31]
HITS update rule for authorities	$x_i := \sum_{j \in I_i} y_j$	[31]
Page-rank update rule with damping factor 0.85	$\rho_i := 0.15 + 0.85 \sum_{j \in I_i} \frac{\rho_j}{\delta_j^-}$	[13]

two HITS-related features and page rank, our criterion for convergence has been that, for all nodes, the two latest feature values differ from each other by some quantity in the interval  $[-10^{-16}, 10^{-16}]$ .

CCD plots for the local features are given in Figure 1 (in-degree, out-degree, and degree), Figure 2 (centralities), and Figure 3 (hub, authority, and page rank). One striking characteristic they all share is that no feature of any library seems to be expressible as a clear power law for any significant number of orders of magnitude. For example, although we have found the in-degree CCD for DLMF to be given approximately by a power-law of  $\alpha = 2.47$ , this seems reasonable only for one order of magnitude (roughly between 10 and 100). In the case of Figure 1, in particular, this widespread absence of a power law works to confirm the expectation that, in such a specialized domain as the five libraries', it is expertise, rather than some popularity-based criterion such as preferential attachment, that guides the establishment of connections.

In Figure 2, the CCD plots for the  $C_i$  and  $G_i$  values share the peculiar property that all nodes are concentrated inside three relatively narrow centrality intervals. For each of the five libraries, first are the sink nodes, those for which  $R_i = O_i = \emptyset$ , having  $C_i = G_i = 0$ . Then comes what in almost all cases is the most densely populated interval. Nodes in this interval have the relatively small centrality values typically associated with relatively large distances to the nodes in  $R_i$ . They are members of the graph's largest so-called in-component, which encompasses the GSCC and all nodes from which at least one directed path leads to the GSCC. This explains the single exception, which once again concerns the small-GSCC graph of the Wikipedia library with only See-also links ( $W'$ ). The third centrality interval contains the remaining nodes and is characterized by centrality values that in almost all cases bespeak relatively small distances to the nodes in  $R_i$ . These nodes lie outside the graph's largest in-component and, once again, the single exception is relative to  $W'$ .

## 5 Local features and GSCC disruption

In the graphs we have been studying, as in all graphs reflecting real-world networks, the existence of the GSCC is merely a matter of observation: we simply look for the graph's strongly connected components and select the largest one. In a more abstract sense, however, random-graph models of networks have been studied for the existence of such components under a growth regime from relative sparseness to relative denseness (that is, as the graph's number of nodes and/or edges is changed so that it becomes denser). Such studies were initiated with the Erdős-Rényi (ER) random graphs [21], which are undirected and characterized by a Poisson distribution of node degrees. Since edges do not have directions in the ER model, one looks for weakly (rather than strongly) connected components, or simply connected components, and for the GCC (rather than the GSCC). It turns out that, increasing  $\delta$  (the mean degree) past 1 as the graph becomes denser gives sudden rise to the GCC as a connected component that for the first time is set apart from the others by virtue of its size [22]. A

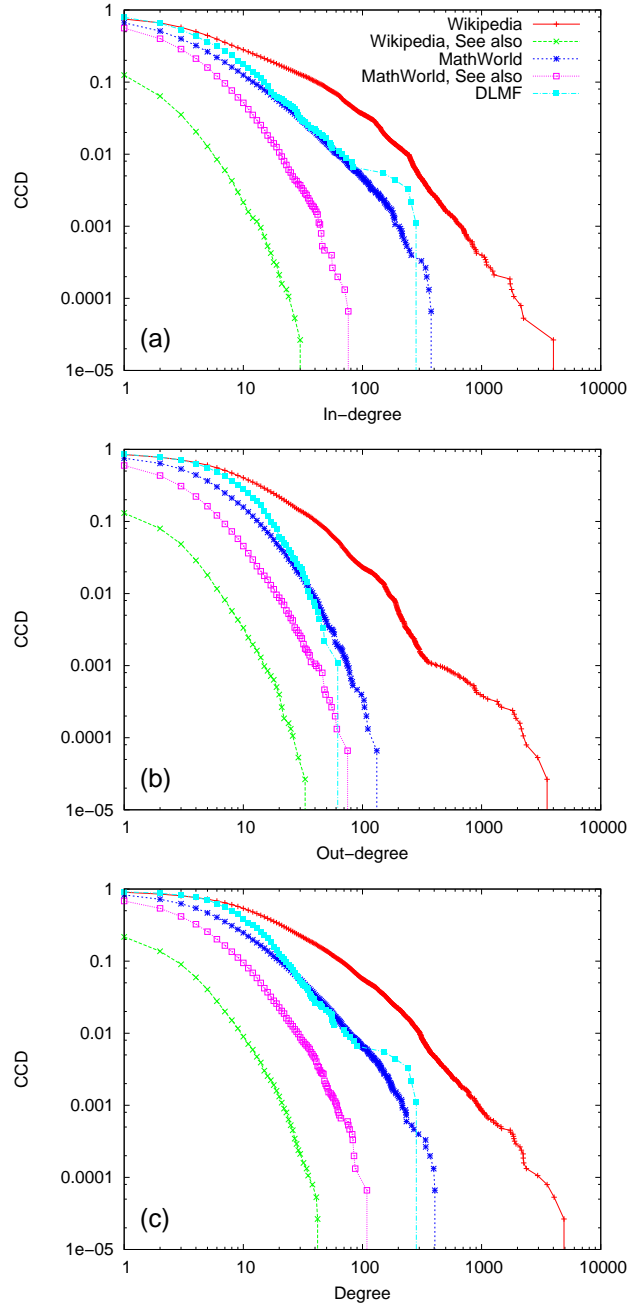


Figure 1: CCD plots for the  $\delta_i^+$  (a),  $\delta_i^-$  (b), and  $\delta_i$  (c) values of  $W$  (Wikipedia),  $W'$  (Wikipedia, See also),  $M$  (MathWorld),  $M'$  (MathWorld, See also), and  $D$  (DLMF).

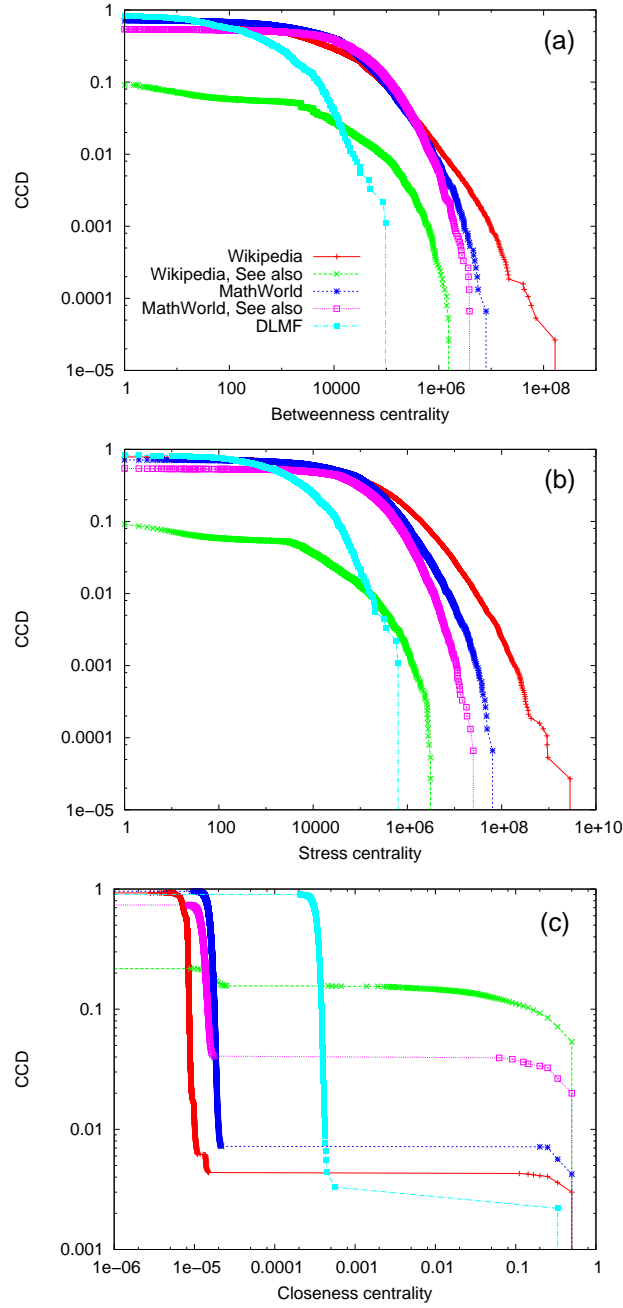


Figure 2: CCD plots for the  $B_i$  (a),  $S_i$  (b),  $C_i$  (c), and  $G_i$  (d) values of  $W$  (Wikipedia),  $W'$  (Wikipedia, See also),  $M$  (MathWorld),  $M'$  (MathWorld, See also), and  $D$  (DLMF).

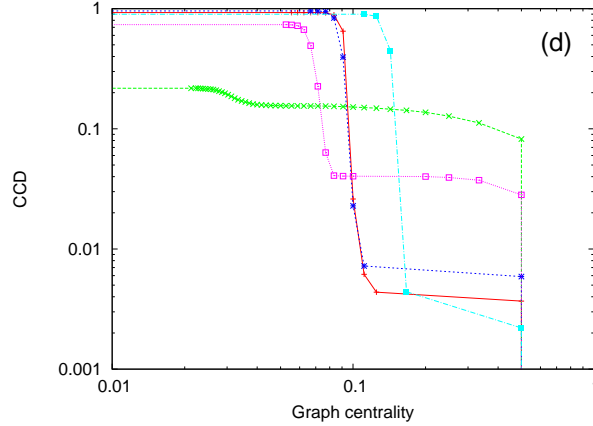


Figure 2: Continued.

similar phenomenon also occurs in many other random-graph models, including their directed variations with regard to the rise of the GSCC [30, 33, 34, 20, 40].

Another similar phenomenon, often called site percolation, is the breakdown of the GCC or GSCC when nodes are continually isolated from the rest of the graph by the removal of all edges incident to them. In the case of ER graphs, for example, the GCC breakdown is expected to happen after a fraction  $1 - 1/\delta$  of the nodes has been randomly isolated [9], provided  $\delta > 1$  to begin with (i.e., provided there really is a GCC initially). Results of this sort have been obtained also for undirected graphs with degrees obeying a scale-free distribution. However, unlike the ER graphs, with their degrees closely clustered about the mean, now there may exist high-degree nodes, so it makes sense to look at targeted as well as random node-isolation processes. As it turns out, for  $\alpha = 2.5$  (which is thought to describe the Internet graph) the GCC is only expected to disappear after at least 99% of the nodes have been randomly isolated, although for relatively small graphs this can be as low as about 80% [18]. Targeting highest-degree nodes first, though, implies that isolating fewer than 20% of the nodes is expected to suffice [3]. We know of no similar studies for directed random-graph models regarding the impact of node isolation on the graph's GSCC. So, despite the figures given above, we are essentially left without any meaningful clue as to what to expect when we conduct node isolation in our five mathematical libraries.

The results we present in this section describe the evolution of  $S$ , the fraction of  $n$  inside the GSCC, as nodes are isolated either randomly or targeting first the non-isolated node for which a specific local feature is highest. In the former case we provide the average value obtained from ten independent trials. As for the local feature in question, we report on all ten discussed in Section 4. In all cases, node isolation is performed until no strongly connected component has more than one node. When isolation stops, then, all remaining nodes are either

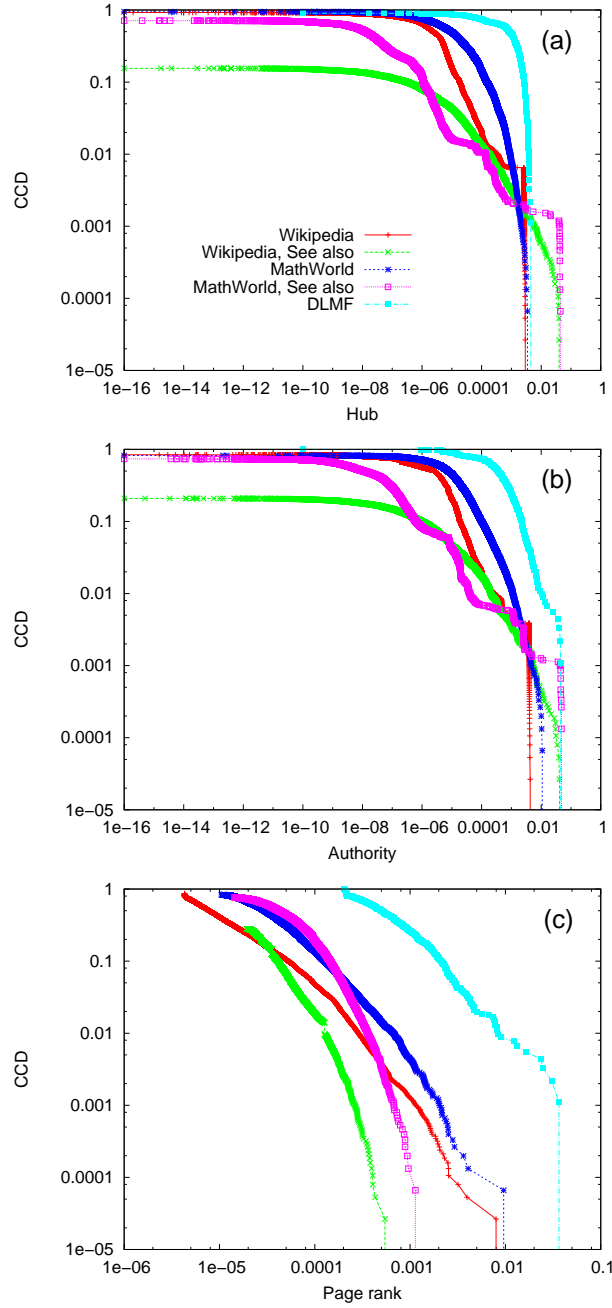


Figure 3: CCD plots for the  $y_i$  (a),  $x_i$  (b), and  $\rho_i$  (c) values of  $W$  (Wikipedia),  $W'$  (Wikipedia, See also),  $M$  (MathWorld),  $M'$  (MathWorld, See also), and  $D$  (DLMF).

isolated (no in- or out-neighbors) or part of an acyclic portion of the current graph.

Our results appear in Figure 4, where the breakdown fractions for random isolation are seen to be in the  $[0.4, 0.6]$  interval for the non-See-also graphs, along with roughly 0.3 for  $M'$  and less than 0.01 for  $W'$ . If we once again except  $W'$ , with its frail GSCC, and maybe  $M'$  as well, we are left with figures that indicate what seem to be quite resilient GSCCs in  $M$ ,  $W$ , and  $D$ . As we turn to the isolation of nodes following one of the local features, the data in Figure 4 reveal that the specific feature in question is practically irrelevant, with the exception of graph centrality and closeness centrality in all cases but that of  $W'$ . These two features are, respectively, the second and third least effective means we have found to break the GSCC (following the random method, which is the least effective). Except for graph and closeness centrality, the data also reveal a breakdown fraction of about 0.2 for  $W$  and  $D$ , then a little above 0.1 for  $M$ , then a little below 0.1 for  $M'$  and, finally, less than 0.01 for  $W'$ . Targeting nodes based on any one of these local features, then, reveals a dividing line between the See-also and non-See-also graphs as well, with  $W'$  once more at the lowest end and  $M'$  in between  $W'$  and the non-See-also group.

## 6 Local features and text search

Google’s search engine grew out of the notion of page rank, one of the ten local features examined in Section 4. Page rank, however, is no more of a node descriptor than any of the other local features, so in principle it is at least conceivable that any of the others might be used instead. We explore such possibility in this section for each of the five directed graphs  $W$ ,  $W'$ ,  $M$ ,  $M'$ , and  $D$ , regarding the search, in their nodes’ texts, for a number of the top keywords in mathematics as reported at the Microsoft Academic Search (MAS) site<sup>5</sup> as of November 1, 2012.

We follow the standard method outlined in [5]. For each graph and local feature, and for a given query (one of the aforementioned keywords), this method begins by identifying a list  $A$  of answer nodes (sorted by nonincreasing feature value) as well as a set  $R$  of relevant nodes. It then proceeds to calculating the well-known Precision and Recall metrics for each  $k = 1, 2, \dots, |A|$ . These are given by the fraction of  $k$  corresponding to the nodes in the size- $k$  prefix of list  $A$  that are also in  $R$  (Precision) and the fraction of  $|R|$  that corresponds to these shared nodes (Recall). Note that, the more relevant nodes are ranked first in  $A$ , the higher Precision values are obtained for a larger stretch of Recall values.

The elements of  $A$  are simply those nodes whose texts contain the keyword in question. As for  $R$ , normally it would be identified by a group of experts. In the absence of one, however, we identify it by resorting to all ten local features, not just the feature that is being analyzed and was used to sort  $A$ , and letting each one “vote” for or against each potential candidate for inclusion in  $R$ . Set

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<sup>5</sup><http://academic.research.microsoft.com/RankList?entitytype=8&topDomainID=15&subDomainID=0>.



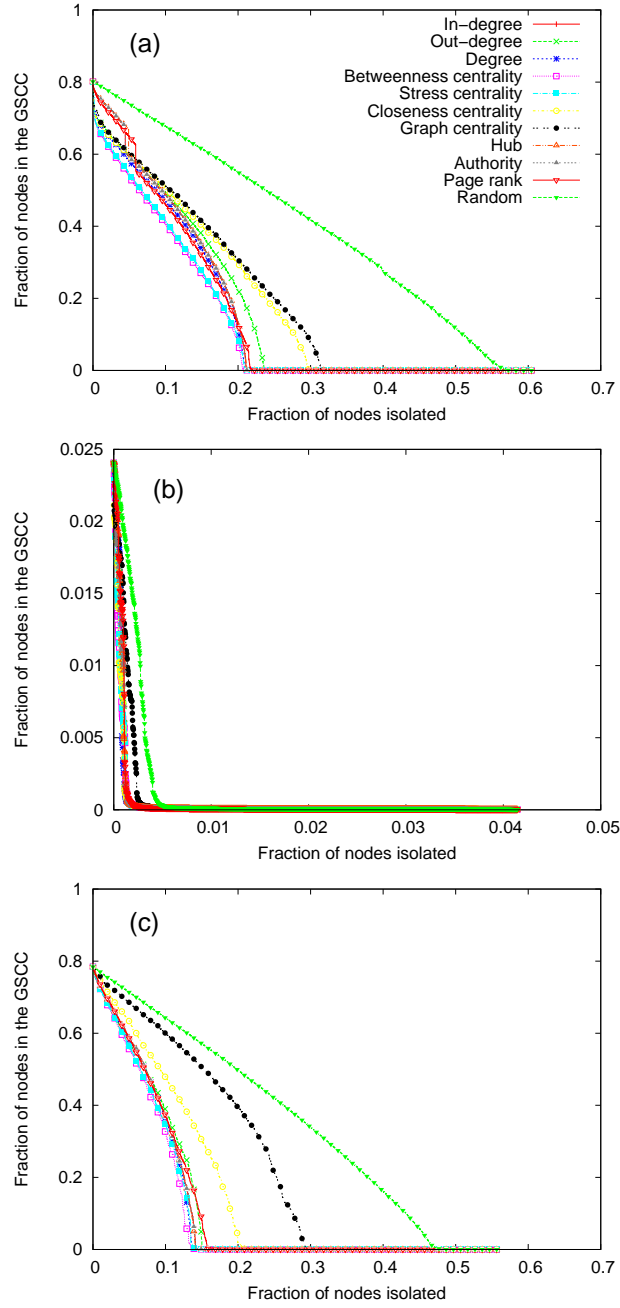


Figure 4: Evolution of  $S$  under node isolation in  $W$  (Wikipedia; a),  $W'$  (Wikipedia, See also; b),  $M$  (MathWorld; c),  $M'$  (MathWorld, See also; d), and  $D$  (DLMF; e).

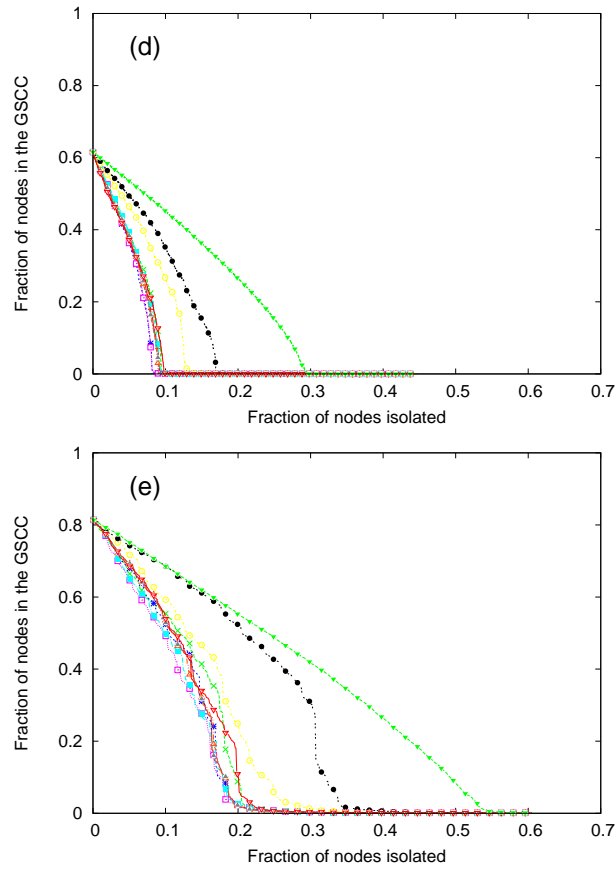


Figure 4: Continued.

$R$ , therefore, is as much a function of the feature used to sort  $A$  as it is of the others. The following steps summarize the construction of set  $R$ :

1. Let  $X$  be the set of nodes in whose texts the desired keyword appears. If  $|X| \leq 10$ , go to Step 5.
2. Create ten sorted lists of the nodes in  $X$ , each by nonincreasing order of one of the ten local features.
3. Let  $Y$  be the set of nodes that appear amid the top ten nodes in a strict majority (i.e., at least six) of the ten lists.
4. Let  $R := Y$  and stop.
5. Let  $R := \emptyset$  and stop.

Note that requiring  $|X| > 10$  for termination to occur in Step 4 is necessary to avoid the trivial case of  $R = X$ , which allows for no discrimination of the local features vis-à-vis one another. When the requirement is not met and termination occurs in Step 5, the query in question is dropped.

Our results, given next, refer to those MAS keywords, out of the top 300, for which the procedure above terminated in Step 4 in our experiments. Whenever such keywords numbered more than 100, we considered only the top 100. As it turns out, we obtained the desired 100 keywords for all graphs but  $D$ , which ended up with only 14 keywords (i.e., only 14 of the 300 keywords were found in more than ten nodes). Figure 5 contains the resulting Precision-Recall plots. They are given as Precision averages relative to eleven Recall intervals, viz.  $[0, 0.1)$ ,  $[0.1, 0.2)$ ,  $\dots$ ,  $[1, 1]$ , plotted respectively at the abscissae  $0, 0.1, \dots, 1$ .

According to the data in Figure 5, in order to search the mathematical portion of Wikipedia through the use of local features based on graph  $W$  it is best to use page rank, followed very closely by either the hub or authority feature. Should the search be based on graph  $W'$ , however, then one should use the hub criterion as the absolute champion. Notice, notwithstanding this, that the use of  $W'$  incurs a loss of Precision of about 10% relative to the use of  $W$  and cannot be recommended on any grounds. Still regarding Wikipedia, our data also indicate that, if one is willing to examine the list  $A$  of answer nodes past the point at which about 70% of the set  $R$  of relevant nodes have been covered, then the nodes' degrees and stress centralities turn out to be the local features to be preferred for sorting  $A$ .

Turning to MathWorld we find a wholly different picture in the data, since now the best local feature to sort  $A$  is the nodes' stress-centrality values as given by graph  $M$ , regardless of how much of  $R$  one is willing to examine. If one is willing to examine no more than about 50% of  $R$ , though, then the nodes' betweenness-centrality values are equally effective. As for using graph  $M'$ , and unlike the case of Wikipedia, only a small loss of Precision is incurred in comparison with  $M$  (about 1 or 2%), but now the local feature of preference to sort  $A$  is betweenness centrality, followed very closely by the nodes' degrees (for examining up to about 60% of  $R$ ).

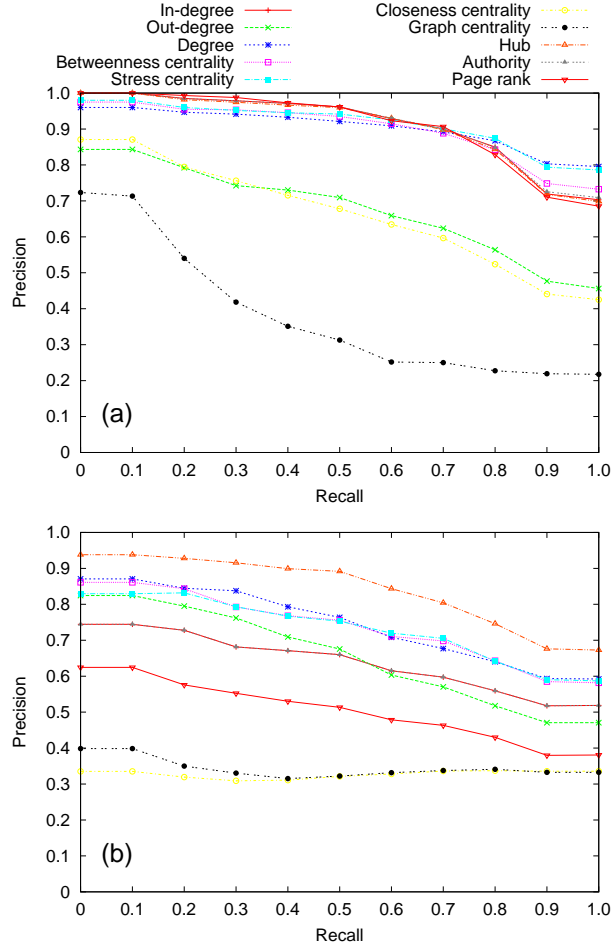


Figure 5: Precision-Recall plots for  $W$  (Wikipedia; a),  $W'$  (Wikipedia, See also; b),  $M$  (MathWorld; c),  $M'$  (MathWorld, See also; d), and  $D$  (DLMF; e).

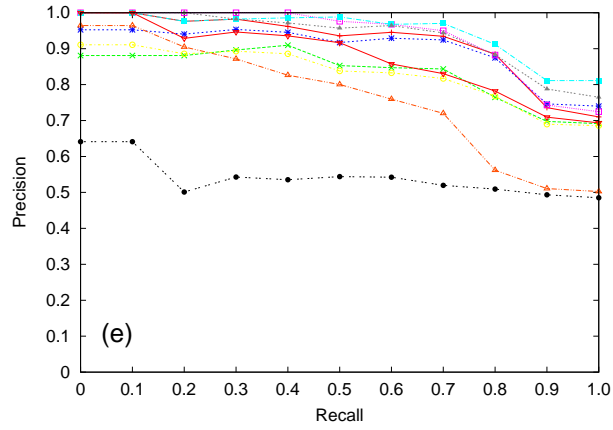
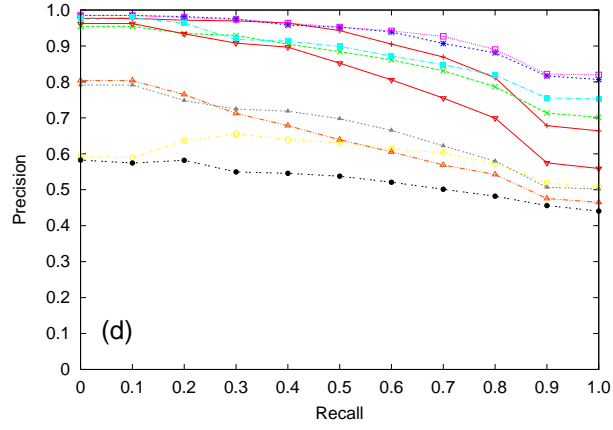
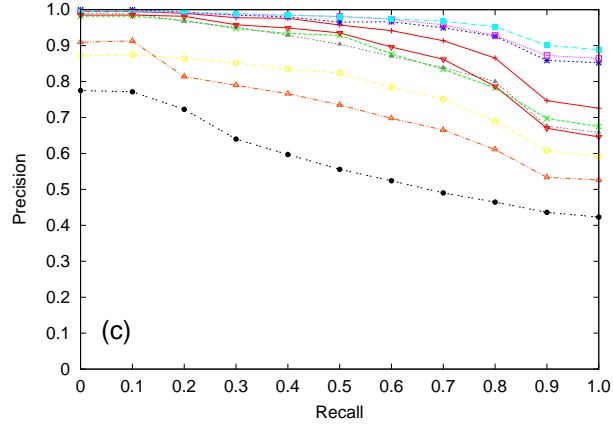


Figure 5: Continued.

As for DLMF, the local feature of choice is once again betweenness centrality (for up to about 40% of  $R$ ), though the nodes' authority values are equally effective (up to about 20% of  $R$ ), and so are the nodes' stress-centrality values and in-degrees (up to about 10% of  $R$ ). Should one be willing to examine about 50% of set  $R$  or more, then stress centrality becomes the local feature to be preferred.

## 7 Conclusions

We have studied three online mathematical libraries, viz. the mathematical portion of Wikipedia, MathWorld, and DLMF, from the perspective of network theory. To this end, we considered directed graphs whose nodes are library pages and whose edges reflect the directed pairing of pages through the links that point from one to another. In the case of Wikipedia and MathWorld, these links come in two clearly identifiable categories (those that are in-text and those in the pages' See-also sections), so we considered two separate graphs for each of these libraries. We focused on both global and local network-theoretic properties of these graphs, aiming at characterizing them, studying their resiliency to the accidental or intentional loss of material, and also assessing how best to perform text search in the pages that their nodes stand for.

Among our key finds are the presence of GSCCs that in most cases encompass node fractions substantially larger than that of the Web, indications of small-world phenomena, practically no signs of relevant assortativity in the linking patterns, and the absence of any clear power laws describing the distributions of local features. We also found that most graphs are quite resilient to the accidental loss of material, though naturally less so when we consider the intentional destruction of pages. As for searching the libraries for the occurrence of specific keywords, only for Wikipedia do the customary criteria of page rank and the HITS-related features perform best. For the smaller MathWorld and DLMF, primacy is taken by local features that hitherto do not appear to have been considered for this purpose, notably stress centrality, betweenness centrality, and the nodes' degrees.

We believe that many of these finds can be attributed to one key distinguishing property of all three libraries. Unlike what happens in several other domains, where such intangibles as affinity or popularity dictate the establishment of connections, in building these libraries what matters is how knowledgeable each contributor is on the core material being treated and on how it relates to the other topics. That this key distinction should surface in the form of measurable effects such as the networks' structural properties and their consequences, and that this should happen despite the typically large number of often independent contributors involved, is quite remarkable.

We finalize with a note on some related work on MathWorld that precedes our own analysis [29]. Such work is based on a December 2008 version of the library, so it predates the one we use by some eight months (cf. Table 1). Despite this relatively short span of intervening time, our graph has 25% more nodes

(about 3000 nodes beyond that work's 12000), so we conjecture that some intermittent failure during the download process may have caused the loss of material. In [29] the authors give the distributions of in- and out-degrees and of betweenness centrality. Despite the considerable difference between the two graphs, our results agree with theirs in that neither in-degrees nor out-degrees are distributed as power laws. Their betweenness-centrality distribution also appears consistent with ours, though they seem to have missed the page for "Triangle," which we find to be one of the top ten for this local feature but they do not. They also discuss clustering, average distance, and assortativity, but the definitions they use for these quantities are not the most commonly used and are incompatible with ours.

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